Hermite variations of the fractional Brownian sheet

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Dedicated to Paul Malliavin

Abstract

We prove central and non-central limit theorems for the Hermite variations of the anisotropic fractional Brownian sheet $W^{\alpha,\beta}$ with Hurst parameter $(\alpha,\beta)\in(0,1)^2$. When $0<\alpha\leq 1-\frac{1}{2q}$ or $0<\beta\leq 1-\frac{1}{2q}$ a central limit theorem holds for the renormalized Hermite variations of order $q\geq 2$, while for $1-\frac{1}{2q}<\alpha,\beta<1$ we prove that these variations satisfy a non-central limit theorem. In fact, they converge to a random variable which is the value of a two-parameter Hermite process at time $(1,1)$.

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1 Introduction

In recent years a lot of attention has been given to the study of the (weighted) power variations for stochastic processes. Let us recall the case of the fractional Brownian motion (fBm). Consider $B^H := (B^H_t)_{t\in[0,1]}$ a fBm with Hurst parameter H in $(0,1)$. Recall that B^H is a centered Gaussian process with covariance $R_H(t,s)=\mathbb{E}(B^H_t B^H_s)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$ for every $s,t\in[0,1]$. It can also be defined as the only self-similar Gaussian process with stationary increments. The weighted Hermite variations of order $q\geq 1$ of B^H are defined as

$$V_N := \sum_{i=1}^{N} f(B^H_{(i-1)/N})H_q(N^H(B^H_{i/N}-B^H_{(i-1)/N})),$$

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where H_q denotes the Hermite polynomial of order q (see Section 2.2) and f is a real-valued deterministic function regular enough. Take for example $q = 2$. Since the second-order Hermite polynomial is $H_2(x) = \frac{x^2-1}{2}$, the latter quantity is equal to $\frac{1}{2}\sum_{i=0}^{N-1} f(B^H_{(i-1)/n})(N^{2H}|B^H_{i/N} - B^H_{(i-1)/N}|^2 - 1)$. The asymptotic behavior of these variations plays an important role in estimating the parameter of the fractional Brownian motion or of other self-similar processes (see e.g. [7] or [24]). Weighted Hermite variations are also crucial in the study of numerical schemes for stochastic differential equations driven by a fBm (see [12]). A full understanding of V_n is given in [3, 8, 9, 21], when $f \equiv 1$ and in [14, 17] for quite general functions f. Let us recall the main results for the case $f \equiv 1$:

- If $0 < H < 1 - 1/(2q)$ then, $\quad N^{-1/2} V_N \xrightarrow[n\to\infty]{law} \mathcal{N}(0, C_H)$,
- If $H = 1 - 1/(2q)$ then, $\quad (\log(N)N)^{-1/2} V_N \xrightarrow[n\to\infty]{law} \mathcal{N}(0, C_H)$,
- If $1 - 1/(2q) < H < 1$ then, $\quad N^{1-2H} V_N \xrightarrow[n\to\infty]{law}$ Hermite r.v.

Here C_H is an explicit positive constant. A Hermite random variable is the value at time 1 of a Hermite process, which is a non-Gaussian self-similar process with stationary increments living in the qth Wiener chaos (see e.g. [14]).
In this paper we use Malliavin calculus and multiple stochastic integrals to study the asymptotic behavior of the non-weighted (*i.e.* $f \equiv 1$) Hermite variations, where the fBm is replaced by a fractional Brownian sheet (fBs), which is a centered Gaussian process $(W^{\alpha,\beta}_{(s,t)})_{(s,t)\in[0,1]^2}$ whose covariance function is the product of the one of a fBm of parameter α in one direction and of the covariance of a fBm of parameter β in the other component. We define the Hermite variations based on the rectangular increments of $W^{\alpha,\beta}$ by, for every $N, M \geq 1$,

$$V_{N,M} := \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} H_q\left(N^\alpha M^\beta \left(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\right)\right).$$

In some sense the fBs is the tensorization of two orthogonal fBm. In the view of results from the one-dimensional case mentioned above it would be natural to expect the limit of the correctly renormalized non-weighted Hermite variations to be the "tensorization" of the limits appearing in the one-parameter case. This actually is true in several cases: when $\alpha, \beta \leq 1 - \frac{1}{2q}$, the limit in distribution of the renormalized sequence $V_{N,}$ is, as expected, a Gaussian random variable and when $\alpha, \beta > 1 - \frac{1}{2q}$ the limit (in L^2 actually) of $V_{N,}$ is the value at time $(1,1)$ of a two-parameter Hermite process (which will be introduced later in the paper). The most interesting and quite unexpected case is when one Hurst parameter is less and the other one is strictly bigger than the critical value $1-\frac{1}{2q}$. It turns out that in this situation the limit in distribution of the renormalized Hermite variations is still Gaussian.
We prove our central limit theorems using Malliavin calculus and the so-called Stein's method on Wiener chaos introduced by Nourdin and Peccati in [15]. Using these results, it is actually possible to measure the distance between the law of an arbitrary random variable

F (differentiable in the sense of the Malliavin calculus) and the standard normal law. This distance can be bounded by a quantity which involves the Malliavin derivative of F. Using these tools and analyzing the Malliavin derivatives of $V_{N,M}$ (which is an element of the q th Wiener chaos generated by the fBs $W^{\alpha,\beta}$) we are able to derive a Berry-Esséen bound in our central limit theorem.

We proceed as follows. In Section 2 we define the Hermite variations of a fBs and give the basic tools of Malliavin calculus for the fractional Brownian sheet needed throughout the paper. The central case is presented in Section 3, whereas Section 4 is devoted to the non-central case. The Appendix contains some auxiliary technical lemmas.

2 Preliminaries

2.1 The fractional Brownian sheet

Several extensions of the fractional Brownian motion have been proposed in the literature as for example the *fractional Brownian field* ([11, 4]), the *Lévy's fractional Brownian field* ([6]) and the *anisotropic fractional Brownian sheet* ([10, 2]), which we consider in this paper. The definitions and properties of this section can be found in [1, 22]. We begin with the definition of the anisotropic fractional Brownian sheet.

Definition 1 (Fractional Brownian sheet) *A fractional Brownian sheet* $(W^{\alpha,\beta}_{s,t})_{(s,t)\in[0,1]^2}$ *with Hurst indices* $(\alpha,\beta)\in(0,1)^2$ *is a centered two-parameter Gaussian process equal to zero on the set*

$$\{(s,t)\in[0,1]^2,\ s=0 \ \text{or}\ t=0\}.$$

For $s_1,t_1,s_2,t_2\in[0,1]$ *the covariance function is given by*

$$\begin{aligned} R^{\alpha,\beta}((s_1,t_1),(s_2,t_2)) &:= \mathbb{E}\left[W^{\alpha,\beta}_{s_1,t_1}W^{\alpha,\beta}_{s_2,t_2}\right] \\ &= K^{\alpha}(s_1,s_2)K^{\beta}(t_1,t_2) \\ &= \frac{1}{2}\left(s_1^{2\alpha}+s_2^{2\alpha}-|s_1-s_2|^{2\alpha}\right)\frac{1}{2}\left(t_1^{2\beta}+t_2^{2\beta}-|t_1-t_2|^{2\beta}\right). \end{aligned}$$

We assume that $(W^{\alpha,\beta}_{s,t})_{(s,t)\in[0,1]^2}$ is defined on a complete probability space $(\Omega,\mathcal{F},P)$ where $\mathcal{F}$ is generated by $W^{\alpha,\beta}$. Let us denote by $\mathcal{H}^{\alpha,\beta}$ the canonical Hilbert space generated by the Gaussian process $W^{\alpha,\beta}$ defined as the closure of the linear span generated by the indicator functions on $[0,1]^2$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,s_1]\times[0,t_1]},\mathbf{1}_{[0,s_2]\times[0,t_2]}\rangle_{\mathcal{H}^{\alpha,\beta}} = R^{\alpha,\beta}((s_1,t_1),(s_2,t_2)).$$

The mapping $\mathbf{1}_{[0,s]\times[0,t]}\mapsto W^{\alpha,\beta}_{s,t}$ provides an isometry between $\mathcal{H}^{\alpha,\beta}$ and the first chaos of $W^{\alpha,\beta}$ denoted by $H_1^{\alpha,\beta}$. For an element φ of $\mathcal{H}^{\alpha,\beta}$ we denote by $W^{\alpha,\beta}(\varphi)$ the image of φ in the space $H_1^{\alpha,\beta}$.

For any γ in $(0,1)$ we denote by $\mathcal{H}^\gamma$ the Hilbert space defined as the closure of the linear span generated by indicator functions on $[0,1]$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,s_1]}, \mathbf{1}_{[0,s_2]} \rangle_{\mathcal{H}^\gamma} = \frac{1}{2}\big(s_1^{2\gamma} + s_2^{2\gamma} - |s_1 - s_2|^{2\gamma}\big).$$

The space $\mathcal{H}^\gamma$ is in fact the canonical Hilbert space generated by the (one-dimensional) fractional Brownian motion with Hurst parameter $\gamma \in (0,1)$. With these notations we will often use the practical relation

$$\langle \mathbf{1}_{[0,s_1]\times[0,t_1]}, \mathbf{1}_{[0,s_2]\times[0,t_2]} \rangle_{\mathcal{H}^{\alpha,\beta}} = \langle \mathbf{1}_{[0,s_1]}, \mathbf{1}_{[0,s_2]} \rangle_{\mathcal{H}^\alpha} \langle \mathbf{1}_{[0,t_1]}, \mathbf{1}_{[0,t_2]} \rangle_{\mathcal{H}^\beta}, \quad \forall (s_1,s_2,t_1,t_2) \in [0,1]^4.$$

More generally, for any two functions $f, g \in \mathcal{H}^{\alpha,\beta}$ such that $\int_{[0,1]^4} |f(u,v)g(a,b)|\, |u - a|^{2\alpha-2}|v-b|^{2\beta-2} \mathrm{d}a\mathrm{d}b\mathrm{d}u\mathrm{d}v < \infty$ we have

$$\langle f, g \rangle_{\mathcal{H}^{\alpha,\beta}} = a(\alpha)a(\beta) \int_{[0,1]^4} f(u,v)g(a,b)|u-a|^{2\alpha-2}|v-b|^{2\beta-2}\mathrm{d}a\mathrm{d}b\mathrm{d}u\mathrm{d}v \tag{1}$$

with $a(\alpha) = \alpha(2\alpha - 1)$. Note finally that we can also give a representation of $(W^{\alpha,\beta}_{s,t})_{(s,t)\in[0,1]^2}$ as a stochastic integral of kernels K^α and K^β with respect to a standard Brownian sheet $(W_{(s,t)})_{(s,t)\in[0,1]^2}$:

$$W^{\alpha,\beta}_{(s,t)} = \int_0^s \int_0^t K^\alpha(s,u)K^\beta(t,v)\, dW_{(u,v)}, \quad (s,t) \in [0,1]^2,$$

where K^α is the usual kernel of the fractional Brownian motion B^α which appears in its expression as a Wiener integral $B^\alpha_t = \int_0^t K^\alpha(t,s)dW_s$ (see e.g. [18] for an explicit definition of this kernels; we will not use it in this paper). Using this representation, Tudor and Viens in [22, 23] have developed a Malliavin calculus with respect to $W^{\alpha,\beta}$.
Let us recall the notion of self-similarity and stationary increments for a two-parameter process (see [2]).

Definition 2 *A two-parameter stochastic process $(X_{s,t})_{(s,t)\in T}$, $T \subset \mathbb{R}^2$ has stationary increments if for every $n \in \mathbb{N}$ and for every $(s_1,t_1), (s_2,t_2), \ldots, (s_n,t_n) \in T$ the law of the vector*

$$(X_{s+s_1,t+t_1}, X_{s+s_2,t+t_2}, \ldots, X_{s+s_n,t+t_n})$$

does not depends on $(s,t) \in T$.

Definition 3 *A two-parameter stochastic process $(X_{s,t})_{(s,t)\in T}$, $T \subset \mathbb{R}^2$ is self-similar with order (α,β) if for any $h,k > 0$ the process $(\widehat{X}_{s,t})_{s,t\in T}$*

$$\widehat{X}_{s,t} := h^\alpha k^\beta X_{\frac{s}{h},\frac{t}{k}}$$

has the same law as the process X.

Note that the fractional Brownian sheet $W^{\alpha,\beta}$ is self-similar and has stationary increments in the sense of Definitions 3 and 2 (see [2]).
Now we present some elements of Malliavin calculus with respect to fractional Brownian sheets and especially the Malliavin integration by parts formula (4).

2.2 Malliavin calculus for the fractional Brownian sheet

We recall some definitions and properties of the Malliavin calculus for the fractional Brownian sheet. For general Gaussian processes these are contained in the framework described in [18].

By $\mathcal{C}_b^\infty(\mathbb{R}^n)$ we denote the space of infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ with bounded partial derivatives. For a cylindrical functional F of the form

$$F = f\left(W^{\alpha,\beta}(\varphi_1), \dots, W^{\alpha,\beta}(\varphi_n)\right), \quad n \geq 1,\ \varphi_1, \dots, \varphi_n \in \mathcal{H}^{\alpha,\beta},\ f \in \mathcal{C}_b^\infty(\mathbb{R}^n), \tag{2}$$

we define the Malliavin derivative DF of F as,

$$DF := \sum_{i=1}^n \partial_i f\left(W^{\alpha,\beta}(\varphi_1), \dots, W^{\alpha,\beta}(\varphi_n)\right)\varphi_i.$$

Furthermore $D : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}, P; \mathcal{H}^{\alpha,\beta})$ is a closable operator and it can be extended to the closure of the Sobolev space $\mathbb{D}^{1,2}$ defined by the functionals F whose norm $\|F\|_{1,2}$ is finite with

$$\|F\|_{1,2}^2 := \mathbb{E}\left[F^2\right] + \mathbb{E}\left[\|DF\|_{\mathcal{H}^{\alpha,\beta}}^2\right].$$

The adjoint operator I_1 of D is called the divergence operator and it is defined by the following duality relationship

$$\mathbb{E}\left[FI_1(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}^{\alpha,\beta}}\right],$$

for F in $\mathbb{D}^{1,2}$ and for u in $\mathcal{H}^{\alpha,\beta}$ such that there exists a constant $c_u > 0$, satisfying

$$\left|\mathbb{E}\left[\langle DG, u\rangle_{\mathcal{H}^{\alpha,\beta}}\right]\right| \leq c_u \|G\|_{L^2(\Omega,\mathcal{F},P)}, \qquad \text{for every functional } G \text{ of the form (2).}$$

Let $n \geq 1$. The nth Wiener chaos $\mathfrak{H}_n$ of $W^{\alpha,\beta}$ is the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\left\{H_n\left(W^{\alpha,\beta}(\varphi)\right),\ \varphi \in \mathcal{H}^{\alpha,\beta},\ \|\varphi\|_{\mathcal{H}^{\alpha,\beta}} = 1\right\}$ where H_n denotes the nth Hermite polynomial

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right).$$

A linear isometry between the symmetric tensor product $\left(\mathcal{H}^{\alpha,\beta}\right)^{\odot n}$ and $\mathfrak{H}_n$ is defined as

$$I_n\left(\varphi^{\otimes n}\right) := n! H_n\left(W^{\alpha,\beta}(\varphi)\right). \tag{3}$$

We conclude this section by the following integration by parts formula:

$$\mathbb{E}\left[FI_n(h)\right] = \mathbb{E}\left[\langle D^n F, h\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes n}}\right], \quad h \in \left(\mathcal{H}^{\alpha,\beta}\right)^{\odot n},\ F \in \mathbb{D}^{n,2}, \tag{4}$$

where $\mathbb{D}^{n,2}$ is the space of functionals F such that $\|F\|_{n,2}$ is finite with

$$\|F\|_{n,2}^2 := \mathbb{E}\left[F^2\right] + \sum_{i=1}^n \mathbb{E}\left[\|D^i F\|_{\mathcal{H}^{\alpha,\beta}}^2\right].$$

2.3 Hermite variations of the fBs

Let $(W^{\alpha,\beta}_{s,t})_{s,t\geq 0}$ be a fractional Brownian sheet with Hurst parameter $(\alpha,\beta)\in(0,1)^2$. We will define the Hermite variations of order $q\geq 1$ of the fractional Brownian sheet by

$$V_{N,M} := \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} H_q\left(N^\alpha M^\beta \left(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\right)\right), \tag{5}$$

where H_q is the Hermite polynomial of order q. Note that

$$\mathbb{E}\left(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\right)^2 = N^{-2\alpha}M^{-2\beta},$$

which explains the appearance of the factor $N^\alpha M^\beta$ in (5): with this factor the random variable $N^\alpha M^\beta\left(W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}}\right)$ has L^2-norm equal to 1.
We will use the notation

$$\Delta i = \left[\frac{i}{N},\frac{i+1}{N}\right] \text{ and } \Delta i,j = \left[\frac{i}{N},\frac{i+1}{N}\right]\times\left[\frac{j}{M},\frac{j+1}{M}\right] = \Delta i\times\Delta j,$$

for $i\in\{0,\dots,N-1\},\ j\in\{0,\dots,M-1\}$. In principle $\Delta i = \Delta i^{(N)}$ depends on N but we will omit the superscript N to simplify the notation. With this notation we can write

$$W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i}{N},\frac{j+1}{M}} - W^{\alpha,\beta}_{\frac{i+1}{N},\frac{j}{M}} + W^{\alpha,\beta}_{\frac{i}{N},\frac{j}{M}} = I_1\left(\mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right) = I_1(\mathbf{1}_{\Delta i,j}) = I_1(\mathbf{1}_{\Delta i\times\Delta j}).$$

Here, and throughout the paper, I_n indicates the multiple integral of order $n>1$ with respect to the fractional Brownian sheet $W^{\alpha,\beta}$. Since for any deterministic function $h\in\mathcal{H}^{\alpha,\beta}$ with norm one we have

$$H_q(I_1(h)) = \frac{1}{q!}I_q(h^{\otimes q}),$$

we derive at

$$V_{N,M} = \frac{1}{q!}\sum_{i=1}^{N}\sum_{j=1}^{M} N^{\alpha q}M^{\beta q} I_q\left(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right).$$

We want to study the limit of the (suitably normalized) sequence $V_{N,M}$ as $N,M\to\infty$. Since this normalization is depending on the choice of α and β, we will normalize it with a function $\varphi(\alpha,\beta,N,M)$.
Let us define

$$\tilde{V}_{N,M} := \frac{1}{q!}\varphi(\alpha,\beta,N,M)\sum_{i=1}^{N}\sum_{j=1}^{M} I_q\left(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right). \tag{6}$$

By renormalization of the sequence $V_{N,M}$ we understand a function $\varphi(\alpha,\beta,N,M)$ to fulfill the property $\mathbb{E}\tilde{V}^2_{N,M} \overset{N,M\to\infty}{\to} 1$.

It turns out that the limit of the sequence $\tilde{V}_{N,M}$ is either Gaussian, or a Hermite random variable, which is the value at time $(1,1)$ of a two-parameter Hermite process.
In the case when $\tilde{V}_{N,M}$ converges to a Gaussian random variable, our proof will be based on the following result (see [15], see also [16]).

Theorem 1 *Let F be a random variable in the qth Wiener chaos. Then*

$$d(F,N) \quad \leq \quad c\sqrt{\mathbb{E}\left(\left(1-q^{-1}\left\|DF\right\|^2_{\mathcal{H}^{\alpha,\beta}}\right)^2\right)} \tag{7}$$

The above inequality still holds true for several distances (Kolmogorov, Wasserstein, total variation or Fortet-Mourier). The constant c is equal to 1 in the case of the Kolmogorov and of the Wasserstein distance, c=2 for the total variation distance and $c=4$ in the case of the Fortet-Mourier distance.

Until the end of this paper d will denote one of the distances mentioned in the previous theorem. We will also assume that $q\geq 2$ because for $q=1$ we have $H_1=x$ and then $V_{N,M}$ is Gaussian; this case is trivial. Our argumentation has the following structure. We first compute the Malliavin derivative (with respect to the fractional Brownian sheet $W^{\alpha,\beta}$) $D\tilde{V}_{N,M}$ and we compute its norm in the space $\mathcal{H}^{\alpha,\beta}$. We will get

$$D\tilde{V}_{N,M} = \frac{1}{(q-1)!}\varphi(\alpha,\beta,N,M)\sum_{i=1}^{N}\sum_{j=1}^{M} I_{q-1}\left(\mathbf{1}^{\otimes q-1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right)\mathbf{1}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]},$$

and

$$\begin{aligned}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} \quad &= \quad \frac{1}{(q-1)!^2}(\varphi(\alpha,\beta,N,M))^2\times\sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\langle\mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\rangle_{\mathcal{H}^{\alpha,\beta}}\\ &\quad\times I_{q-1}\left(\mathbf{1}^{\otimes q-1}_{\Delta i,j}\right)I_{q-1}\left(\mathbf{1}^{\otimes q-1}_{\Delta i',j'}\right).\end{aligned}$$

The product formula for multiple integrals (see [18], Chapter 1) reads

$$\begin{aligned}I_{q-1}\left(\mathbf{1}^{\otimes q-1}_{\Delta i,j}\right)I_{q-1}\left(\mathbf{1}^{\otimes q-1}_{\Delta i',j'}\right) \quad &= \quad \sum_{p=0}^{q-1}p!(C^p_{q-1})^2\langle\mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\rangle^p_{\mathcal{H}^{\alpha,\beta}}\\ &\quad\times I_{2q-2-2p}\left(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\right),\end{aligned}$$

where $C^p_{q-1} := \binom{q-1}{p}$ for $q\geq 2, p\leq q-1$ and $f\tilde{\otimes}g$ denotes the symmetrization of the function $f\otimes g$. Hence, we have

$$\begin{aligned}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} \quad &= \quad \frac{1}{(q-1)!^2}(\varphi(\alpha,\beta,N,M))^2\times\sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\sum_{p=0}^{q-1}\langle\mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}}p!\\ &\quad\times(C^p_{q-1})^2 I_{2q-2-2p}\left(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\right).\end{aligned}$$

Let us isolate the term $p = q-1$ in the above expression. In this case $2q-2-2p = 0$ and this term gives the expectation of $\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}}$.

$$
\begin{aligned}
&\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} && (8)\\
= \; & \frac{1}{(q-1)!^2}(\varphi(\alpha,\beta,N,M))^2 \\
&\times \sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\sum_{p=0}^{q-2}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}} p!(C^p_{q-1})^2 I_{2q-2-2p}\left(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde{\otimes}\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\right) \\
&+\frac{1}{(q-1)!}(\varphi(\alpha,\beta,N,M))^2\sum_{i,i'=0}^{N-1}\sum_{j,j'=0}^{M-1}\langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{q}_{\mathcal{H}^{\alpha,\beta}} =: T_1 + T_2. && (9)
\end{aligned}
$$

The term T_2 is a deterministic term which is equal to $\mathbb{E}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}}$.
With the correct choice of the normalization we will show that T_2 is converging to q as N, M goes to infinity and T_1 converges to zero in L^2 sense. Using Theorem 1 we will prove the convergence to a standard normal random variable of $\tilde{V}_{N,M}$ and we give bounds for the speed of convergence. The distinction between the two cases (when the limit is normal and when the limit is non-Gaussian) will be made by the term T_1: it converges to zero if $\alpha \le 1-\frac{1}{2q}$ or $\beta \le 1-\frac{1}{2q}$, while for $\alpha,\beta > 1-\frac{1}{2q}$ this term converges to a constant.
Let us first discuss the normalization $\varphi(\alpha,\beta,N,M)$ and the convergence of T_2 in the following lemma. Given two sequences of real numbers $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$, we write $a_n \trianglelefteq b_n$ for $\sup_{n\ge1}\frac{|a_n|}{|b_n|} < \infty$.

Lemma 1 *Let T_2 be as in (9). Then $q^{-1}T_2 \overset{N,M\to\infty}{\to} 1$ for the following choices of φ:*

1) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha s_\beta}}N^{\alpha q-1/2}M^{\alpha q-1/2}$, *if* $0<\alpha,\beta<1-\frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{2q\alpha-2q+1} + M^{-1} + M^{2q\beta-2q+1}$,

2) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha \iota_\beta}}N^{\alpha q-1}M^{q-1}(\log M)^{-1/2}$, *if* $0<\alpha<1-\frac{1}{2q}$, $\beta = 1-\frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{2q\alpha-2q+1} + (\log M)^{-1}$,

3) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{\iota_\alpha \iota_\beta}}N^{q-1}(\log N)^{-1/2}M^{q-1}(\log M)^{-1/2}$, *if* $\alpha=\beta=1-\frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq (\log N)^{-1} + (\log M)^{-1}$,

4) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha \kappa_\beta}}N^{\alpha q-1/2}M^{q-1}$, *if* $0<\alpha<1-\frac{1}{2q}$, $\beta>1-\frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-1} + N^{-2q\alpha+2q-1} + M^{-2q\beta+2q-1}$

5) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{\iota_\alpha \kappa_\beta}} N^{q-1}(\log N)^{-1/2} M^{q-1}$, *if* $0 < \alpha = 1 - \frac{1}{2q}$, $\beta > 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq (\log(N)^{-1} + M^{-2q\beta+2q-1}$.

6) $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{\kappa_\alpha \kappa_\beta}} N^{q-1} M^{q-1}$, *if* $\alpha > 1 - \frac{1}{2q}$, $\beta > 1 - \frac{1}{2q}$ *and* $q^{-1}T_2 - 1 \trianglelefteq N^{-2q\alpha+2q-1} + M^{-2q\beta+2q-1}$.

where $s_.$, $\iota_.$ *and* $\kappa_.$ *are defined in Lemma 2 in the Appendix.*

Proof: Using the properties of the scalar product in Hilbert spaces we have

$$\begin{aligned} T_2 &= \frac{1}{(q-1)!}(\varphi(\alpha,\beta,N,M))^2 \times \sum_{i,i'=0}^{N-1} \sum_{j,j'=0}^{M-1} \langle \mathbf{1}_{\Delta i,j}(\cdot), \mathbf{1}_{\Delta i',j'}(\cdot)\rangle^q_{\mathcal{H}^{\alpha,\beta}} \\ &= \frac{1}{(q-1)!}(\varphi(\alpha,\beta,N,M))^2 \times \left(\sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}(\cdot), \mathbf{1}_{\Delta i'}(\cdot)\rangle^q_{\mathcal{H}^\alpha} \right) \times \left(\sum_{j,j'=0}^{M-1} \langle \mathbf{1}_{\Delta j}(\cdot), \mathbf{1}_{\Delta j'}(\cdot)\rangle^q_{\mathcal{H}^\beta} \right). \end{aligned}$$

The result follows then from Lemma 2 in the Appendix. ■

Remark 1 *As mentioned above,* $T_2 = \mathbb{E}\|D\tilde{V}_{N,M}\|_{\mathcal{H}^{\alpha,\beta}}$. *On the other hand, we also have*

$$qT_2 = \mathbb{E}\tilde{V}^2_{N,M}.$$

Indeed, this is true because for every multiple integral $F = I_q(f)$, *it holds that* $\mathbb{E}F^2 = q\mathbb{E}\|DF\|^2_{\mathcal{H}^{\alpha,\beta}}$.

3 The Central Limit Case

We will prove that for every $\alpha,\beta \in (0,1)^2 \setminus \left(1 - \frac{1}{2q}, 1\right)^2$ a Central Limit Theorem holds, where $\tilde{V}_{N,M}$ was defined in (6). Using the Stein's method we also give the Berry-Esséen bounds for this convergence.

Theorem 2 *(Central Limits)*
Let $\tilde{V}_{N,M}$ *be defined by (6). For every* $(\alpha,\beta) \in (0,1)^2$, *we denote by* $c_{\alpha,\beta}$ *a generic positive constant which depends on* α,β,q *and on the distance* d *and which is independent of* N *and* M. *We have*

1) If $0 < \alpha,\beta < 1 - \frac{1}{2q}$, *then* $\tilde{V}_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha s_\beta}} N^{\alpha q - 1/2} M^{\alpha q-1/2}$. *In addition*

$$d(\tilde{V}_{N,M},\mathcal{N}) \leq c_{\alpha,\beta}\sqrt{N^{-1} + N^{2\alpha-2} + N^{2\alpha q-2q+1} + M^{-1} + M^{2\beta-2} + M^{2\beta q-2q+1}}.$$

2) If $0<\alpha<1-\frac{1}{2q}$ *and* $\beta=1-\frac{1}{2q}$*, then* $\tilde V_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{s_\alpha \iota_\beta}}N^{\alpha q-1}M^{q-1}(\log M)^{-1/2}$*. In addition*

$$d(\tilde V_{N,M},\mathcal{N})\le c_{\alpha,\beta}\sqrt{N^{-1}+N^{2\alpha-2}+N^{2\alpha q-2q+1}+(\log M)^{-1}}.$$

3) If both $\alpha=\beta=1-\frac{1}{2q}$*, then* $\tilde V_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{\iota_\alpha \iota_\beta}}N^{q-1}(\log N)^{-1/2}M^{q-1}(\log M)^{-1/2}$*. In addition*

$$d(\tilde V_{N,M},\mathcal{N})\le c_{\alpha,\beta}\sqrt{\log N^{-1}+\log M^{-1}}.$$

4) If $\alpha<1-\frac{1}{2q}$ *and* $\beta>1-\frac{1}{2q}$*, then* $\tilde V_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{s_\alpha \kappa_\beta}}N^{\alpha q-1/2}M^{q-1}$*. In addition*

$$d(\tilde V_{N,M},\mathcal{N})\le c_{\alpha,\beta}\sqrt{N^{-1}+N^{2\alpha-2}+N^{2\beta q-2q+1}+M^{2\beta q-2q+1}}.$$

5) If $\alpha=1-\frac{1}{2q}$ *and* $\beta>1-\frac{1}{2q}$*, then* $\tilde V_{N,M}$ *converges in law to a standard normal r.v.* $\mathcal{N}$ *with normalization* $\varphi(\alpha,\beta,N,M)=\sqrt{\frac{q!}{\iota_\alpha \kappa_\beta}}N^{q-1}(\log N)^{-1/2}M^{q-1}$*. In addition*

$$d(\tilde V_{N,M},\mathcal{N})\le c_{\alpha,\beta}\sqrt{\log(N)^{-1}+M^{2\beta q-2q+1}}.$$

Proof: Recall that

$$\|D\tilde V_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}} \quad = \quad : T_1+T_2,$$

where the summands T_1 and T_2 are given as in (9). We apply Lemma 1 to see that $1-q^{-1}T_2$ converges to zero as N,M goes to infinity.
Let us show that T_1 is converging to zero in $L^2(\Omega)$. We use the orthogonality of the iterated integrals to compute

$$\begin{aligned}
\mathbb{E}T_1^2 &= \frac{1}{(q-1)!^4}(\varphi(\alpha,\beta,N,M))^4\sum_{i,i',k,k'=0}^{N-1}\sum_{j,j',\ell,\ell'=0}^{M-1}\sum_{p=0}^{q-2}(p!)^2\\
&\quad (C^p_{q-1})^4\langle \mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}}\langle \mathbf{1}_{\Delta k,\ell}(\cdot),\mathbf{1}_{\Delta k',\ell'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}}\\
&\quad \times\mathbb{E}\left[I_{2q-2-2p}\left(\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde\otimes\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'}\right)I_{2q-2-2p}\left(\mathbf{1}^{\otimes q-1-p}_{\Delta k,\ell}\tilde\otimes\mathbf{1}^{\otimes q-1-p}_{\Delta k',\ell'}\right)\right]\\
&= \frac{1}{(q-1)!^4}(\varphi(\alpha,\beta,N,M))^4\sum_{i,i',k,k'=0}^{N-1}\sum_{j,j',\ell,\ell'=0}^{M-1}\sum_{p=0}^{q-2}(p!)^2\\
&\quad (C^p_{q-1})^4\langle \mathbf{1}_{\Delta i,j}(\cdot),\mathbf{1}_{\Delta i',j'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}}\langle \mathbf{1}_{\Delta k,\ell}(\cdot),\mathbf{1}_{\Delta k',\ell'}(\cdot)\rangle^{p+1}_{\mathcal{H}^{\alpha,\beta}}\\
&\quad \times(2q-2-2p)!\langle\mathbf{1}^{\otimes q-1-p}_{\Delta i,j}\tilde\otimes\mathbf{1}^{\otimes q-1-p}_{\Delta i',j'},\mathbf{1}^{\otimes q-1-p}_{\Delta k,\ell}\tilde\otimes\mathbf{1}^{\otimes q-1-p}_{\Delta k',\ell'}\rangle_{\mathcal{H}^{\alpha,\beta}}.
\end{aligned}$$

Now, let us discuss the tensorized terms. We use the fact that

$$
\begin{aligned}
& \langle \mathbf{1}_{\Delta i,j}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta i',j'}^{\otimes q-1-p}, \mathbf{1}_{\Delta k,\ell}^{\otimes q-1-p} \tilde{\otimes} \mathbf{1}_{\Delta k,\ell'}^{\otimes q-1-p} \rangle_{\mathcal{H}^{\alpha,\beta}} \\
= & \sum_{a+b=q-1-p;c+d=q-1-p} \langle \mathbf{1}_{\Delta i,j}, \mathbf{1}_{\Delta k,\ell} \rangle_{\mathcal{H}^{\alpha,\beta}}^{a} \langle \mathbf{1}_{\Delta i,j}, \mathbf{1}_{\Delta k',\ell'} \rangle_{\mathcal{H}^{\alpha,\beta}}^{b} \\
& \times \langle \mathbf{1}_{\Delta i',j'}, \mathbf{1}_{\Delta k,\ell} \rangle_{\mathcal{H}^{\alpha,\beta}}^{c} \langle \mathbf{1}_{\Delta i',j'}, \mathbf{1}_{\Delta k',\ell'} \rangle_{\mathcal{H}^{\alpha,\beta}}^{d} \\
= & \sum_{a+b=q-1-p;c+d=q-1-p} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^{\alpha}}^{a} \langle \mathbf{1}_{\Delta j}, \mathbf{1}_{\Delta \ell} \rangle_{\mathcal{H}^{\beta}}^{a} \\
& \times \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^{\alpha}}^{b} \langle \mathbf{1}_{\Delta j}, \mathbf{1}_{\Delta \ell'} \rangle_{\mathcal{H}^{\beta}}^{b} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^{\alpha}}^{c} \langle \mathbf{1}_{\Delta j'}, \mathbf{1}_{\Delta \ell} \rangle_{\mathcal{H}^{\beta}}^{c} \\
& \times \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^{\alpha}}^{d} \langle \mathbf{1}_{\Delta j'}, \mathbf{1}_{\Delta \ell'} \rangle_{\mathcal{H}^{\beta}}^{d}
\end{aligned}
$$

(we recall that $\mathbf{1}_{\Delta i} := \mathbf{1}_{[\frac{i}{N}, \frac{i+1}{N}]}$). Therefore we finally have

$$
\begin{aligned}
\mathbb{E}T_1^2 = & \frac{1}{(q-1)!^4} (\varphi(\alpha,\beta,N,M))^4 \sum_{p=0}^{q-2} (C_{q-1}^{p})^4 (p!)^2 \sum_{a+b=q-1-p;c+d=q-1-p} \\
& a_N(p,\alpha,a,b,c,d) b_M(p,\beta,a,b,c,d),
\end{aligned}
$$

with

$$
\begin{aligned}
a_N(p,\alpha,a,b,c,d) = & \sum_{i,i',k,k'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^{\alpha}}^{a} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^{\alpha}}^{b} \\
& \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k} \rangle_{\mathcal{H}^{\alpha}}^{c} \langle \mathbf{1}_{\Delta i'}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^{\alpha}}^{d} \\
& \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^{\alpha}}^{p+1} \langle \mathbf{1}_{\Delta k}, \mathbf{1}_{\Delta k'} \rangle_{\mathcal{H}^{\alpha}}^{p+1}
\end{aligned}
$$

and $b_M(p,\beta,a,b,c,d)$ similarly defined. We apply Lemma 3 stated in the Appendix to the terms a_N and b_M to conclude the convergence of T_1 to zero. Hence, $\mathbb{E}[(1-q^{-1}\|D\tilde{V}_{N,M}\|^2_{\mathcal{H}^{\alpha,\beta}})^2] = q^{-2}\mathbb{E}[|T_1|^2]+(1-q^{-1}T_2)^2$ which converges to zero for $\alpha \leq 1-\frac{1}{2q}$ or $\beta \leq 1-\frac{1}{2q}$. The bounds on the rate of convergence are given by the Lemmas 3 and 1. Using Theorem 1, the conclusion of the theorem follows. ■

The fact that the term T_1 converges to zero makes the difference between the situations treated in the above theorem and the non-central limit case proved in the next section.

4 The Non Central Limit Theorem

We will assume throughout this section that the Hurst parameters α, β satisfy

$$1 > \alpha, \beta > 1 - \frac{1}{2q}.$$

We will study the limit of the sequence $\tilde{V}_{N,M}$ given by the formula (6) with the renormalization factor φ from Lemma 1, point 6. Let us denote by $h_{N,M}$ the kernel of the random variable $\tilde{V}_{N,M}$ which is an element of the qth Wiener chaos, i.e.

$$h_{N,M} = \frac{1}{q!}\varphi(\alpha,\beta,N,M)\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}.$$

We will prove that $(h_{N,M})_{N,M\geq 1}$ is a Cauchy sequence in the Hilbert space $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}$. Using relation (1), we obtain

$$\begin{aligned}\langle h_{N,M}, h_{N',M'}\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}} &= \frac{1}{q!^2}\varphi(\alpha,\beta,N,M)\varphi(\alpha,\beta,N',M')\\ &\times(\alpha(2\alpha-1))^q\sum_{i=0}^{N-1}\sum_{i'=0}^{N'-1}\left(\int_{\frac{i}{N}}^{\frac{i+1}{N}}\int_{\frac{i'}{N}}^{\frac{i'+1}{N}}|u-v|^{2\alpha-2}dudv\right)^q\\ &\times(\beta(2\beta-1))^q\sum_{j=0}^{M-1}\sum_{j'=1}^{M'-1}\left(\int_{\frac{j}{M}}^{\frac{j+1}{M}}\int_{\frac{j'}{M'}}^{\frac{j'+1}{M'}}|u-v|^{2\beta-2}dudv\right)^q\end{aligned}$$

and this converges to (see also [5] or [24])

$$c_2(\alpha,\beta)\frac{1}{q!^2}(\alpha(2\alpha-1))^q(\beta(2\beta-1))^q\int_0^1\int_0^1|u-v|^{(2\alpha-2)q}dudv\int_0^1\int_0^1|u-v|^{(2\beta-2)q}dudv,$$

where $c_2(\alpha,\beta)=\frac{q!}{\kappa_\alpha\kappa_\beta}$. The above constant is equal to

$$c_2(\alpha,\beta)\frac{1}{q!^2}(\alpha(2\alpha-1))^q(\beta(2\beta-1))^q\frac{1}{(\alpha q-q+1)(2\alpha q-2q+1)}\frac{1}{(\beta q-q+1)(2\beta q-2q+1)}.$$

Remark 2 *Note that the above constant is actually* $\frac{1}{q!}$.

It follows that the sequence $h_{N,M}$ is Cauchy in the Hilbert space $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}$ and as $N,M\to\infty$ it has a limit in $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}$ denoted by $\mu^{(q)}$. In the same way, the sequence

$$h_{N,M}(t,s) = \frac{1}{q!}\varphi(\alpha,\beta,N,M)\sum_{i=0}^{[(N-1)t]}\sum_{j=0}^{[(M-1)s]}\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}$$

is Cauchy in $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}$ for every fixed s,t and it has a limit in this Hilbert space which will be denoted by $\mu^{(q)}_{s,t}$. Notice that $\mu^{(q)}=\mu^{(q)}(1,1)$ and that $\mu^{(q)}$ is a normalized uniform measure on the set $([0,t]\times[0,s])^q$.

Definition 4 *We define the Hermite sheet process of order q and with Hurst parameters $\alpha, \beta \in (0,1)$, denoted by $(Z_{t,s}^{(q)})_{s,t\in[0,1]}$, by*

$$Z_{t,s}^{(q),\alpha,\beta} := Z_{t,s}^{(q)} = I_q(\mu_{s,t}^{(q)}), \quad \forall s,t \in [0,1].$$

The previous computations lead to the following theorem.

Theorem 3 *Let $\tilde{V}_{N,M}$ be given by (6) with the function φ defined in Lemma 1, point 6. Consider the Hermite sheet introduced in Definition 4. Then for $q \geq 2$ it holds*

$$\lim_{N,M\to\infty} \mathbb{E}[|\tilde{V}_{N,M} - Z|^2] = 0,$$

where $Z := Z_{1,1}^{(q)}$.

Proof: Note that $\frac{1}{q!}\mathbb{E}[|\tilde{V}_{N,M} - Z|^2] = \|h_{N,M}\|^2_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} + \|\mu^{(q)}\|^2_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}} - 2\langle h_{N,M}, \mu^{(q)}\rangle_{(\mathcal{H}^{\alpha,\beta})^{\otimes q}}$. The computations of the beginning of this section complete the proof. ■

Let us prove below some basic properties of the Hermite sheet.

Proposition 1 *Let us consider the Hermite sheet $(Z_{s,t}^{(q)})_{s,t\in[0,1]}$ from Definition 4. We have the following:*

a) The covariance of the Hermite sheet is given by

$$\mathbb{E}Z_{s,t}^{(q)} Z_{u,v}^{(q)} = R_{q(\alpha-1)+1}(s,u) R_{q(\beta-1)+1}(t,v).$$

Consequently, it has the same covariance as the fractional Brownian sheet with Hurst parameters $q(\alpha-1)+1$ and $q(\beta-1)+1$.

b) The Hermite process is self-similar in the following sense: for every $c, d > 0$, the process

$$\hat{Z}_{s,t}^{(q)} := (Z^{(q)})_{cs,dt}$$

has the same law as $c^{q(\alpha-1)+1} d^{q(\beta-1)+1} Z_{s,t}^{(q)}$.

c) The Hermite process has stationary increments in the sense of Definition 2.

d) The paths are Hölder continuous of order (α', β') with $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$.

Proof: Let f be an arbitrary function in $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}$. It holds that

$$
\begin{aligned}
& \langle h_{N,M}(t,s), f\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}} \\
= \quad & c_2(\alpha,\beta)^{-1/2}\frac{N^{q-1}M^{q-1}}{q!}\sum_{i=0}^{[(N-1)t]}\sum_{j=0}^{[(M-1)s]}\langle \mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}, f\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}} \\
= \quad & a(\alpha)^q a(\beta)^q c_2(\alpha,\beta)^{-1/2}\frac{N^{q-1}M^{q-1}}{q!}\sum_{i=0}^{[(N-1)t]}\sum_{j=0}^{[(M-1)s]} \\
& \int_{[0,1]^{2q}} dx_1..dx_q dy_1..dy_q f\left((x_1,y_1),..,(x_q,y_q)\right)\times\int_{[\frac{i}{N},\frac{i+1}{N}]^q} da_1..da_q\int_{[\frac{j}{M},\frac{j+1}{M}]^q} db_1..db_q \\
& \times\prod_{k=1}^{q}|a_k-x_k|^{2\alpha-2}\prod_{k=1}^{q}|b_k-y_k|^{2\beta-2} \\
\overset{N,M\to\infty}{\to} \quad & a(\alpha)^q a(\beta)^q c_2(\alpha,\beta)^{-1/2}\frac{1}{q!}\int_0^t da\int_0^s db\int_{[0,1]^{2q}} dx_1..dx_q dy_1..dy_q \\
& \times f\left((x_1,y_1),..,(x_q,y_q)\right)\prod_{k=1}^{q}|a-x_k|^{2\alpha-2}\prod_{k=1}^{q}|b-y_k|^{2\beta-2}.
\end{aligned}
$$

By applying the above formula for $f=\mu^{(q)}_{u,v}$ and using the fact that

$$
\mathbb{E}\left(Z^{(q)}_{s,t}Z^{(q)}_{u,v}\right)=q!\langle\mu^{(q)}_{s,t},\mu^{(q)}_{u,v}\rangle_{\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes q}},
$$

we obtain the point a).
Concerning b), let us denote by

$$
H_{N,M}(t,s)=\frac{1}{q!}\varphi(\alpha,\beta,N,M)\sum_{i=0}^{[(N-1)t]}\sum_{j=0}^{[(M-1)]s} I_q\left(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right).
$$

We know that

$$
H_{cN,dM}(t,s)\overset{N,M\to\infty}{\to} Z^{(q)}_{t,s} \tag{10}
$$

in $L^2(\Omega)$ for every $s,t\in[0,1]$. But

$$
\begin{aligned}
H_{cN,dM}(t,s) &= \frac{c_2(\alpha,\beta)^{-1/2}}{(cN)^{1-(1-\alpha)q}(dM)^{1-(1-\beta)q}}\sum_{i=0}^{[(N-1)t]}\sum_{j=0}^{[(M-1)s]} I_q\left(\mathbf{1}^{\otimes q}_{[\frac{i}{N},\frac{i+1}{N}]\times[\frac{j}{M},\frac{j+1}{M}]}\right) \\
&= \frac{1}{(c)^{1-(1-\alpha)q}(d)^{1-(1-\beta)q}}H_{N,M}(t,s)\overset{N,M\to\infty}{\to} Z^{(q)}_{t,s}. \qquad (11)
\end{aligned}
$$

The point b) follows easily from (10) and (11).
Point c) is a consequence of the fact that the fractional Brownian sheet has stationary increments in the sense of Definition 2 while point d) can be easily proved by using Kolmogorov

continuity criterion together with points b) and c) above (see also Section 4, page 35-36 in [2]). ■

5 Appendix

We recall the following two technical lemmas which have been proved in [15] and [5].

Lemma 2 *Let γ in $(0,1)$ and q be an integer with $q \geq 2$. We set*

$$r_\gamma(z) := \frac{1}{2}\left(|z+1|^{2\gamma} + |z-1|^{2\gamma} - 2|z|^{2\gamma}\right), \quad z \in \mathbb{Z}.$$

We have:

(i) If $0 < \gamma < 1 - \frac{1}{2q}$, then

$$\lim_{N\to\infty} N^{2\gamma q-1} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q = \sum_{r\in\mathbb{Z}} r_\gamma(z)^q =: s_\gamma,$$

and $|N^{2\gamma q-1} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q - s_\gamma| \trianglelefteq N^{-1} + N^{2q\gamma-2q+1}$.

(ii) If $\gamma = 1 - \frac{1}{2q}$, then

$$\lim_{N\to\infty} \log(N)^{-1} N^{2q-2} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q = 2\left(\frac{(2q-1)(q-1)}{2q^2}\right)^q =: \iota_\gamma,$$

and $|\log(N)^{-1} N^{2q-2} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q - \iota_\gamma| \trianglelefteq \log(N)^{-1}$.

(iii) If $\gamma > 1 - \frac{1}{2q}$, then

$$\lim_{N\to\infty} N^{2q-2} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q = \frac{\gamma^q(2\gamma-1)^q}{(\gamma q - q + 1)(2\gamma q - 2q + 1)} =: \kappa_\gamma,$$

and $|N^{2q-2} \sum_{i,i'=0}^{N-1} \langle \mathbf{1}_{\Delta i}, \mathbf{1}_{\Delta i'} \rangle_{\mathcal{H}^\gamma}^q - \kappa_\gamma| \trianglelefteq N^{2q-1-2\gamma q}$.

Proof: The first two claims can be found respectively in [15, p. 102], [5, p. 491-492]. For the third part we define $f_N := N^{q-1} \sum_{k=0}^{N-1} \mathbf{1}_{[\frac{k}{N}, \frac{k+1}{N}]}^{\otimes q}$. Then f_N is a Cauchy sequence in $(\mathcal{H}^\gamma)^{\otimes q}$ with limit f and $\|f\|_{(H^\gamma)^{\otimes q}}^2 = \kappa_\gamma$. For the rate of convergence we have

$$\begin{aligned} \|f_N\|_{(H^\gamma)^{\otimes q}}^2 - \|f\|_{(H^\gamma)^{\otimes q}}^2 &= \|f_N - f\|_{(H^\gamma)^{\otimes q}}^2 + 2\langle f_N - f, f \rangle_{(H^\gamma)^{\otimes q}} \\ &\leq \|f_N - f\|_{(H^\gamma)^{\otimes q}}^2 + 2\|f_N - f\|_{(H^\gamma)^{\otimes q}}^2 \|f\|_{(H^\gamma)^{\otimes q}}^2. \end{aligned}$$

Refer to [5, Proposition 3.1] to see the details and to get that the order is $O(N^{2q-1-2\gamma q})$ (a direct argument as in the proof of the next lemma can be also employed).

■ We also state the following estimates which have been obtained respectively in [15, p. 102, p. 104] and in [5, p. 491-492].

Lemma 3 *Let γ in $(0,1)$. We set q,p,a,b,c,d integers such that: $q\geq 2$, $p\in\{0,\dots,q-2\}$ and $a+b=c+d=q-1-p$. We have:*

(i) If $0<\gamma<1-\frac{1}{2q}$, then

$$\begin{aligned} & N^{4q\gamma-2}\sum_{i,i',k,k'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta k},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{a}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{b} \\ & \qquad \times\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{c}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{d} \\ \trianglelefteq \ & N^{-1}+N^{2\gamma-2}+N^{2\gamma q-2q+1}. \end{aligned}$$

(ii) If $\gamma=1-\frac{1}{2q}$, then

$$\begin{aligned} & \frac{N^{4q-4}}{\log(N)^2}\sum_{i,i',k,k'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta k},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{a}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{b} \\ & \qquad \times\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{c}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{d} \\ \trianglelefteq \ & \log(N)^{-1}. \end{aligned}$$

(iii) If $\gamma>1-\frac{1}{2q}$, then

$$\begin{aligned} & N^{4q-4}\sum_{i,i',k,k'=0}^{N-1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta k},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{p+1}\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{a}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{b} \\ & \qquad \times\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k}\rangle_{\mathcal{H}^\gamma}^{c}\langle \mathbf{1}_{\Delta i'},\mathbf{1}_{\Delta k'}\rangle_{\mathcal{H}^\gamma}^{d} \\ \trianglelefteq \ & 1. \end{aligned}$$

Proof: The first point is proved in [15, p. 102, 105]. The point (ii) is done in [5, p. 491-492]. The last case can be treated in the following way. The quantity $\langle \mathbf{1}_{\Delta i},\mathbf{1}_{\Delta i'}\rangle_{\mathcal{H}^\gamma}$ is equivalent with a constant times $N^{-2\gamma}|i-i'|^{2\gamma-2}$ and the sum appearing in (iii) is then equivalent to

$$\begin{aligned} & N^{4q-4}N^{-4\gamma q}\sum_{i,i',k,k'=0}^{N-1}|i-i'|^{(2\gamma-2)(p+1)}|k-k'|^{(2\gamma-2)(p+1)} \\ & \times|i-k|^{(2\gamma-2)a}|i'-k'|^{(2\gamma-2)b}|i'-k|^{(2\gamma-2)c}|i-k'|^{(2\gamma-2)d} \\ = \ & N^{-4}\sum_{i,i',k,k'=0}^{N-1}N^{-2q(2\gamma-2)}|i-i'|^{(2\gamma-2)(p+1)}|k-k'|^{(2\gamma-2)(p+1)} \\ & \times|i-k|^{(2\gamma-2)a}|i'-k'|^{(2\gamma-2)b}|i'-k|^{(2\gamma-2)c}|i-k'|^{(2\gamma-2)d}, \end{aligned}$$

for N large enough and this is a Riemann sum which converges to a constant. ■

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